International Journal Of Advanced Research In Engineering Technology & Sciences
ISSN: 2394-2819
July- 2015 Volume 2, Issue-7
Email: editor@ijarets.org www.ijarets.org

A MATHEMATICAL STUDY OF VARIOUS NON-LINEAR DIOPHANTINE EQUATIONS

Dr.Kapil Kumar Bansal
Head, Research&Publication
SRM University, NCR Campus
Ghaziabad

Dr. Mukesh Sharma
Dept. of Mathematics
RSS PG College Pilakhuwa
Hapur

ABSTRACT:
A method of finding integer solutions of a Linear Diophantine equation using continued fractions is well known. Number Theory is that branch of Mathematics which deals with the properties of integers, more specifically the properties of positive integers. It has always occupied a unique position in the world of Mathematics. This is due to the historical importance of the subject. It is one of the few disciplines having demonstrable results. Since classical antiquity nearly every century has witnessed new and fascinating discoveries relating to the properties of numbers. There are so many non-linear Diophantine equations. In this paper, an attempt will be made to discuss some more non-linear Diophantine equations.

KEY WORDS*:* Non-linear diophantine

THE DIOPHANTINE EQUATION:

$$x^2 + 7^{2k+1} = y^n:$$

Cohn (1993) solved the Diophantine equation $x^2 + 5 = y^n$. Arif, et al. (1998) solved the equation $x^2 + 3^{2k+1} = y^n$ completely. Fadwa, S., Muriefah, A. and Arif, A. (1999) solved the equation $x^2 + 5^{2k+1} = y^n$. They showed that the equation has no integral solution in *x*, *y* for $n \geq 3, k \geq 0$. Ramrnujan (1913) discussed the Diophantine equation $x^2 + 7 = 2^n$ and conjectured that this equation has only five solutions. The generalized Ramanujan's equation $x^2 + p = 2^n$, p is a prime number, has been discussed. In this section, the Diophantine equation by Ramanujan has been generalized to $x^2 + 7^{2k+1} = y^n$. For *k*=0 and *y*=2 the equation reduces to the Ramanujan's Diophantine equation.

If we put *x*=13, *k*=1, *y*=2 and *n*=9 then the equation $x^2 + 7^{2k+1} = y^n$ is satisfied. Thus (*x*,*k*,*y*,*n*)= (13,1,2,9) is a solution of the given Diophantine equation.

THEOREM 1:

If *x* is even and *y* is odd in the Diophantine equation $x^2 + 7^{2k+1} = y^n$ then $x^2 \equiv 0 \pmod 8$ or $x^2 \equiv 4 \pmod 8$ according as *x*=4*m*, *m* is an integer or *x*=2*m*, *m* is an odd integer and $y^n \equiv y \pmod 8$ or $y^n \equiv 1 \pmod 8$ according as *n* is odd or even.

Proof. If *x* is an even number and is of the form *x*=4*m*, *m* is any integer then x^2 is of the form $16m^*$, m^* is a positive integer, which is divisible by 8. Therefore $x^2 \equiv 0 \pmod 8$. If *x* is even and is of the form *x=2m*, *m* is an odd integer then x^2 is of the form $4m^*$, m^* is an odd integer, then obviously the remainder is 4 when x^2 is divided by 8. Therefore $x^2 \equiv 4 \pmod 8$. Further if we take *y*=3 and *n*=3 then $y^n = 3^3 = 27 \equiv 3 \pmod 8$. Similarly if we take *y*=5 and *n*=3 then $y^n = 5^3 = 125 \equiv 5 \pmod 8$. Thus it can be shown that $y^n \equiv y \pmod 8$

when y is odd and n is odd. If we take y=3 and n=4 then $y^n = 3^4 = 81 \equiv 1(\bmod 8)$. If we take y=5 and n=4 then $y^n = 5^4 = 625 \equiv 1(\bmod 8)$. Thus it can be shown that $y^n \equiv 1(\bmod 8)$ when y is odd and n is even.

THEOREM 2:

The possible solution (x,y) of the Diophantine equation $x^2 + 7^{2k+1} = y^n$ can not be simultaneously even or odd for any integral value of k and n.

Proof. If x is even then obviously left hand side of the given Diophantine equation is odd and therefore right hand side can not be even. This implies that y cannot be even. If x is odd then left hand side is obviously even and therefore right hand side can not be odd. This implies that y can not be odd. Thus the possible solution of the given Diophantine equation can not be simultaneously even or odd for any integral values of k and n.

THEOREM 3:

If x is odd and y is even in the Diophantine equation $x^2 + 7^{2k+1} = y^n$ then $x \equiv 1(\bmod 8)$, $x^2 + 7^{2k+1} \equiv 0(\bmod 8)$ and $y \equiv 0(\bmod 8)$ for $n \geq 3$.

Proof. If we take x=3 and k=1 then $x^2 = 3^2 = 9 = 1(\bmod 8)$ and $x^2 + 7^{2k+1} = 3^2 + 7^3$ =9+343=352==0(mod8). Similarly if we take x=5 and k=1 then $x^2 = 5^2 = 25 = 1(\bmod 8)$ and $x^2 + 7^{2k+1} = 5^2 + 7^3$ =25 +343=368=0(mod8).

In this way it can be shown that if x is an odd integer then $x \equiv 1(\bmod 8)$, $x^2 + 7^{2k+1} \equiv 0(\bmod 8)$. If y is even, say, 2k and $n \geq 3$ then $y^n = 8k^* \equiv 0(\bmod 8)$.

THE DIOPHANTINE EQUATION

$$2^n + px^2 = y^p\text{:}$$

LE MAOHUA (1995) discussed the Diophantine equation $2^n + px^2 = y^p$ for p>3. He proved the following two results: (i) The equation has no positive integral solution (x,y,n) if gcd(x,y)=1. (ii) The equation has no positive integral solution (x,y,n) if p is not congruent to (mod8).In the given Diophantine equation $2^n + px^2 = y^p$, p is a prime number. For a given value of p we have to find the positive integral values of x, y and n satisfying the given equation. We consider two values of p ,namely, 3 and 1.

For p=3 we see that x=y=n=4 satisfy the given Diophantine equation $2^n + px^2 = y^p$. Thus (x,y,n)=(4,4,4) is the solution of the Diophantine equation $2^n + 3x^2 = y^3$.

For p=7 we see that x=y=4 and n=2 satisfy the given Diophantine equation. Thus (x,y,n)=(4,4,2) is the solution of the Diophantine equation $2^n + 7x^2 = y^7$.

THEOREM 4:

The possible solution (x,y) of the Diophantine equation $2^n + px^2 = y^p$ will either be both even or both odd for any odd prime p and any integer n.

Proof. Let p be an odd prime and n is any integer. If x is even then obviously the left hand side of the given Diophantine equation is an even number. Therefore right hand side of the equation is also an even number. This implies that y is even. Similarly if x is an odd integer then the left hand side of the given Diophantine equation is an odd integer. Therefore the right hand side is also an odd integer. This implies that y is an odd integer. Thus the solutions x and y of the given Diophantine equation are both even or both are odd integers for a prime p and any integer n.

THE DIOPHANTINE EQUATION

$$x^3 + y^3 + z^3 - 3xyz = o\textbf{:}$$

Here we have to obtained the positive integral values of x, y and z satisfying the given Diophantine equation. Since the factors of left hand side of the given equation are given by

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz),$$

The given equation implies that

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)=0.$$

Since x, y and z are positive the above equation implies that $x^2 + y^2 + z^2 - xy - yz - xz = 0$.

For positive integral solutions, this equation is satisfied for $x=y=z=n$ where n is a positive integer. Thus $(x, y, z) = (n, n, n)$ is the required solution of the given Diophantine equation.

THE DIOPHANTINE EQUATION

$$x^2 + p^q = q^p\textbf{:}$$

Here we have to find the possible solution of the Diophantine equation $x^2 + p^q = q^p$ where p and q are prime numbers. For p=2 and q=3 we see that x=1 satisfies the given Diophantine equation. Thus (x,p,q)=(1,2,3) is the required solution of the given Diophantine equation.

THE DIOPHANTINE EQUATION

$$x^2 = p^{q_1} + p^{q_2}\textbf{:}$$

Here we have to find the possible solution of the Diophantine equation $x^2 = p^{q_1} + p^{q_2}$ where p, q_1 and q_2 are prime numbers. If we take p=3, q_1=2 and q_2=3 then x=6 satisfies the given Diophantine equation. Thus (x, p, q_1, q_2)=(6,3,2,3) is the required solution of the given Diophantine equation.

THE DIOPHANTINE EQUATION

$$x^2 = \sum_{r=0}^{n} q^r\textbf{:}$$

Here we have to find the possible solution of the Diophantine equation $x^2 = \sum_{r=0}^{n} q^r$ where q is a prime number and n is any positive integer.

If we take q=3 then there exist n=1 and x=2 which satisfy the given Diophantine equation. Thus (x,n,q)=(2,1,3) is the required solution of the given Diophantine equation.

For q=3 there exist n=4 and x=11 also which satisfy the given Diophantine equation. Thus (x,n,q)=(11,4,3) is also a solution of the given Diophantine equation.

THE DIOPHANTINE EQUATION

$$x^2 = 1 + \sum_{r=0}^{n-1} q^r\textbf{:}$$

For q=2 the given Diophantine equation takes the form $x^2 = 1 + \sum_{r=0}^{n-1} 2^r$. Now it can be shown that x=$2^{\frac{n}{2}}$ satisfies this equation if n is an even integer. Thus (x,q)=($2^{\frac{n}{2}}$,2) is the required solution of the given Diophantine equation.

THE DIOPHANTINE EQUATION

$$x^2 = \sum_{r=1}^{n} r^q :$$

Here we have to solve the given Diophantine equation $x^2 = \sum_{r=1}^{n} r^q$ where q is a prime number and n is any positive integer. If q=3 then $x = \frac{n(n+1)}{2}$ satisfies the given Diophantine equation $x^2 = \sum_{r=1}^{n} r^q$. Thus $x = \frac{n(n+1)}{2}$ is the required solution of the given equation for q=3.

THE DIOPHANTINE EQUATION

$$x^2 = n^p + n^q + 1:$$

Here we have to find the possible solution of the Diophantine equation $x^2 = n^p + n^q + 1$ where p and q are the prime numbers and n is a positive integer. For p=3 and q=2 we see that x=9 and n=4 satisfy the given Diophantine equation. Thus (x,n,p,q)=(9,4,3,2) is the required solution of the given equation.

THE DIOPHANTINE EQUATION

$$x^2 = p^4 + p^3 + 1:$$

Here we have to find the possible solution of the Diophantine equation $x^2 = p^4 + p^3 + 1$ where p is a prime number. If we put p=2 then x=5 satisfies the given Diophantine equation. Thus (x,p)=(5,2) is the required solution of the given Diophantine equation.

LECTURE HIGHLIGHTS, FURTHER READING, EXTRA PROBLEMS:

1. Diophantine equations are equations with integer coefficients for which a solution is sought where all unknown variables take integer values. Diophantine equations are difficult to solve.
2. Problem solving strategy: When faced with a very difficult problem, it may help to try a simplification of the problem first.
3. Problem solving strategy: It helps to classify large problems into smaller pieces, and attack each piece separately.
4. Coordinates give a relation between equations and curves. Using geometric methods may produce revealing solutions to algebraic problems.
5. Further reading: Rational Points on Elliptic Curve, by J.H. Silverman and J. Tate, part of the Undergraduate Text in Mathematics book series.

REFERENCES:

1. Ming-Guang Leu & Guan-Wei Li (2003): The Diophantine Equation $2x^2 + 1 = 3^n$. *Proc. Amer. Math. Soc.* 131, 3643-3646.
2. Ramanujan, S. (1913): Question 469, *J. Indian Math. Soc.* 5, 59.
3. Dickson,.T,E., History of the Theory ofNun6ers, volume (lI), Diophantine Analysis, Reprinted, New York(I 966).
4. Schmidt, W.M., Diophantine Approximations and Diophantine Equations, Lecture Notes in Mathematics, 2000.
5. Arif, A. Fadwa, S. Abu Muriefah (1998): The Diophantine Equation $x^2 + 3^m = y^n$. *Inter. J. Math. Sci.* 21, 619-630.
6. Cohn, H.E. (1993): The Diophantine Equation $x^2 + c = y^n$. *Acta Arith.* 65, 367-381.
7. Fadwa, S., Muriefah, A. and Arif, A. (1999): The Diophantine Equation $x^2 + 5^{2k+1} = y^n$.*Ind. Jour. Pure Appl. Math.* 30(3), 229-231.
8. Le Maohua (1995): On the Diophantine Equation $2^n + px^2 = y^p$. *Proc. Amer. Math. Soc.* 123, 321-326.